CONSTRAINED AND UNCONSTRAINED PDE'S FOR VECTOR IMAGE RESTORATION

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ABSTRACT

The restoration of noisy and blurred scalar images has been widely studied, and many algorithms based on variational or stochastic formulations have tried to solve this ill-posed problem [2, 4, 10, 7, 33, 20, 19, 18, 22, 1, 26, 24, 9, 28, 29, 6, 30, 35, 37]. However, only few methods exist for multichannel/color images ([7, 29, 16, 36]). Here, we propose a new *vector image restoration PDE* which removes the noise and enhances blurred vector contours, thanks to a vector generalisation of scalar Φ-function diffusions and shock filters. A local and geometric approach is proposed, which uses pertinent vector informations. Finally, we extend this equation to *constrained norm evolutions*, in order to restore direction fields and chromaticity noise on color images.

1. PRINCIPLE OF ANISOTROPIC DIFFUSION

We consider a *scalar image* $I(M) : \Omega \rightarrow \mathbb{R} \;\; (\Omega \in \mathbb{R}^2)$.
Scalar image restoration using Φ-functions classically consists in minimising the following functional :

$$\min_I \int_\Omega \frac{\alpha}{2}\,(I - I_0)^2 + \Phi(\|\nabla I\|)\,d\Omega$$

where $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ is a regularisation function that penalizes high gradients, while preserving edges. The minimisation can be performed via the corresponding *anisotropic PDE evolution*, coming from the Euler-Lagrange equations :

$$\frac{\partial I}{\partial t} = \alpha(I_0 - I) + \operatorname{div}\left(\frac{\Phi'(\|\nabla I\|)}{\|\nabla I\|}\,\nabla I\right)$$

which can be also written as :

$$\frac{\partial I}{\partial t} = \frac{\Phi'(\|\nabla I\|)}{\|\nabla I\|}\,I_{\xi\xi} + \Phi''(\|\nabla I\|)\,I_{\eta\eta} + \alpha(I_0 - I)$$

where $I_{\eta\eta} = \frac{\partial^2 I}{\partial \eta^2} = \nabla(\nabla I.\eta).\eta$ with $\eta = \frac{\nabla I}{\|\nabla I\|}$ and $\xi = \eta^\perp$.

In [19], this expression was interpreted as *two directional 1D heat flows* with different diffusion intensities :

$$c_{\eta\eta} = \Phi''(\|\nabla I\|) \quad \text{and} \quad c_{\xi\xi} = \frac{\Phi'(\|\nabla I\|)}{\|\nabla I\|}$$

in the corresponding directions η and ξ. A diffusion Φ-function must verify these natural properties :

- When $\|\nabla I\| \simeq 0$, the local geometry is flat and doesn't contain any edges, the diffusion must be isotropic :

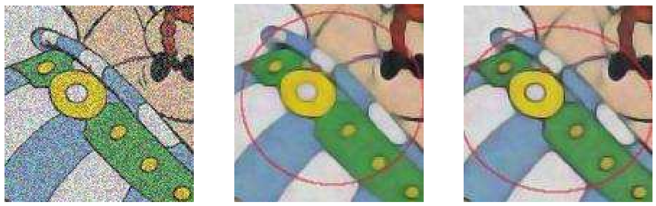
$$c_{\eta\eta} \simeq c_{\xi\xi} \simeq 1 \implies \frac{\partial I}{\partial t} \simeq I_{\xi\xi} + I_{\eta\eta} = \Delta I$$

- When $\|\nabla I\| \gg 0$, the current point may be located on an edge, the diffusion must be anisotropic (oriented by the edge) :

$$c_{\xi\xi} \gg c_{\eta\eta} \implies \frac{\partial I}{\partial t} \simeq c_{\xi\xi}\,I_{\xi\xi}$$

Many Φ-functions were proposed in the literature : Total variation [27], Perona & Malik [24], Geman & McClure [31], Green [13], Hebert-Leahy [14],...
In [19], the authors also proposed to fix directly the smoothing intensities : $c_{\eta\eta} = g_\tau(\|\nabla I\|)$ (decreasing function) and $c_{\xi\xi} = 1$. It ensures a permanent noise removal, but tends to smooth sharp corners.
Geometrically speaking, a PDE restoration process must adapt its diffusion behaviour to the *local geometry of the image*. For the scalar case, this geometry is given by an edge indicator $N(I) = \|\nabla I\|$, and the associated directions η and ξ, respectively orthogonal and parallel to the edges. A vector image diffusion process needs to define such equivalent vector attributes : a *vector gradient norm* $N(\mathbf{I})$, and the corresponding smoothing directions η, ξ for the whole image components, taking the coupling into account. Using a channel by channel approach is then useless : each channel of the image evolves with different smoothing directions

and intensities. The diffusion is not coherent with a *vector geometry* and edges tend to be smoothed (Fig.1).



a) noisy image *b) decoupled PDEs* *c) coupled PDEs*

Fig. 1. Channel by channel approach vs coupled PDEs, on a noisy color image (in the RGB space).

This paper is organised as follow : We first show how to define a local vector geometry, using the classic Di Zenzo method [38], then we compare and interpret some recent vector diffusion PDEs. This comparison yields a new geometric and intuitive vector restoration PDE (eq.(7)) We finally extend this idea to norm constrained evolutions, and propose some results.

2. DEFINING A VECTOR GEOMETRY

Now, we are interested in *vector images* $\mathbf{I}(M) : \mathbb{R}^2 \to \mathbb{R}^n$. I^i denotes the i^{th} image channel ($1 \leq i \leq n$). We want to define a vector gradient norm $N(\mathbf{I})$ and variation directions η and ξ, corresponding to a local vector geometry.

2.1. First approach : scalar conversion

The first idea is to find a function $f : \mathbb{R}^n \to \mathbb{R}$ so that the image $f(\mathbf{I})$ is representative of the human perception of vector edges (for instance, $f = L^*$, the luminance, for color images). Then η can be chosen to be the direction of $\nabla f(\mathbf{I})$, and $N(\mathbf{I}) = \|\nabla f(\mathbf{I})\|$.

The choice of such functions is not an easy task ! However, there are mathematically no functions detecting all possible vector variations. For the color example, it wouldn't be able to detect iso-luminance contours.

2.2. Differential geometry of surfaces

Di Zenzo [38] considers a vector image $\mathbf{I}$ as a $2D \to 3D$ surface, and looks for the local variations of $\|d\mathbf{I}\|^2$:

$$\|d\mathbf{I}\|^2 = \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix}^T \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix}$$

with

$$g_{ij} = \frac{\partial \mathbf{I}}{\partial xi} \cdot \frac{\partial \mathbf{I}}{\partial xj}$$

The two eigenvalues of (g_{ij}) are the extremum of $\|d\mathbf{I}\|^2$ and the orthogonal eigenvectors η,ξ are the corresponding variation directions :

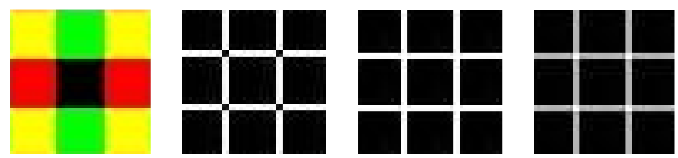
$$\left\{ \begin{array}{l} \lambda_{+/-} = \frac{g_{11}+g_{22}\pm\sqrt{(g_{11}-g_{22})^2+4\ g_{12}^2}}{2} \\ \eta = \frac{1}{2}\ \arctan \frac{2\ g_{12}}{g_{11}-g_{22}} \\ \xi = \eta + \frac{\pi}{2} \end{array} \right. \tag{1}$$

Then, several *vector gradient norms* $N(\mathbf{I})$ can be defined :

- In [29], the authors uses a decreasing function $f(\lambda_+ - \lambda_-)$ to weight their diffusion PDE. It can be seen as a function of a vector variation norm $N(I) = \sqrt{\lambda_+ - \lambda_-}$. Note that this norm fails to detect corners where $\lambda_+ = \lambda_-$ (see the checkboard intersections in Fig.2).
- In [32] and [7], the norm $N(\mathbf{I}) = \sqrt{\lambda_+ + \lambda_-}$ is proposed for a global minimisation process, but can be also used as a local norm definition. It is very easy to compute, since

  $$N(\mathbf{I})^2 = \lambda_+ + \lambda_- = \sum_{k=1}^{n} \|\nabla I^k\|^2$$

  Note that this norm gives more importance to certain corners (but not all) (Fig.2).
- We propose to use $N(\mathbf{I}) = \sqrt{\lambda_+}$, as a direct extension of the gradient norm definition : the value of maximum variation. It doesn't give more or less importance to corners.



Color image $\sqrt{\lambda_+ - \lambda_-}$ $\sqrt{\lambda_+}$ $\sqrt{\lambda_+ + \lambda_-}$

Fig. 2. Differences between vector variation norms

It is worth to mention the work of Kimmel-Malladi-etal [16], which consider a n-D vector image as a surface embedded in a n+2 dimension space. They introduce the induced metric $(g^*_{i,j})$ in a Polyakov functional minimisation, in order to construct a scale space of the vector images. This metric is directly linked to the Di Zenzo approach :

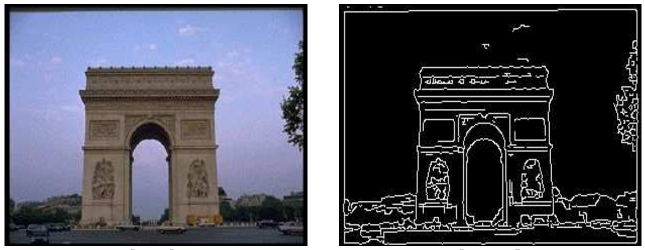
$$g^*_{i,j} = \delta_{i,j} + g_{i,j} \quad \text{where} \quad \delta_{i,j} = \left\{ \begin{array}{ll} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{array} \right.$$

One may note that the corresponding eigenvalues and eigenvector directions are given by :

$$\left\{ \begin{array}{l} \theta^*_+ = \theta_+ \\ \lambda^*_+ = 1 + \lambda_+ \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \theta^*_- = \theta_- \\ \lambda^*_- = 1 + \lambda_- \end{array} \right.$$

These expressions show the similarity between the two approaches.

The Di Zenzo equations define then a pertinent vector geometry with a variation norm $N(\mathbf{I})$ and corresponding directions ξ and η, which can be used in the restoration process to take the coupling between image channels into account. Color edge detection is a direct application of the vector gradient norm definition : One just has to look for the local maxima of $N(\mathbf{I})$ in the η direction (fig.3).



Original color image *Color edges*

Fig. 3. Color edge detector : Thresholded local maxima of $\sqrt{\lambda_+}$ in the η direction

3. DIFFUSION EQUATIONS

We analyse now some proposed vector diffusion equations ([29, 7]), in order to introduce our approach and propose an original and efficient vector diffusion PDE. Comparisons results on synthetic images are shown in the end of this section.

3.1. Sapiro-Ringach's vector diffusion PDE

In [29], the authors propose this anisotropic *vector diffusion PDE* :

$$\frac{\partial \mathbf{I}}{\partial t} = g(\lambda_+ - \lambda_-)\ \mathbf{I}_{\xi\xi} \tag{2}$$

where $g(.)$ is a positive decreasing function with

$$\lim_{s \to +\infty} g(s) = 0$$

and $\mathbf{I}_{\xi\xi} = \frac{\partial^2 \mathbf{I}}{\partial \xi^2}$ (ξ is found with the Di Zenzo calculus).

It was a first step in viewing the importance of coupling in a diffusion process. The diffusion factor $g(\lambda_+ - \lambda_-)$ and the smoothing direction ξ contain informations of coupling between vector components. A vector geometry is taken into account : At a given point, all channels evolve in the same direction and with the same intensity. Edges are then not smoothed (but are not perfectly detected, as described in section 2.2).
Anyway, few problems remain :

- Along very high gradient edges ($N(\mathbf{I}) \gg 0$), smoothing may be weak and doesn't remove the noise : $\frac{\partial \mathbf{I}}{\partial t} \simeq \vec{0}$ (the choice of a function g which doesn't decrease too fast is primordial here).
- In homogenous regions ($N(\mathbf{I}) \simeq 0$), the image pixels diffuses only in the direction ξ, *which is very sensitive to the noise when the geometry is flat !* : $\frac{\partial \mathbf{I}}{\partial t} \simeq c_{\xi\xi}\ \mathbf{I}_{\xi\xi}$. Undesirable texture effects may appear in these regions, because of the uni-directional diffusion.
- No data attachment term : the PDE evolution must be stopped before convergence for a good result.

3.2. Blomgren's TV_{nm} diffusion equation

As defined in [7], the $TV_{n,m}$ diffusion PDE with a component by component writing style is :

$$\frac{\partial I^i}{\partial t} = \frac{TV_{n,1}(I^i)}{TV_{n,m}(\mathbf{I})}\ div\left(\frac{\nabla I^i}{\|\nabla I^i\|}\right) + \alpha\ (I_0^i - I^i) \tag{3}$$

with

$$TV_{n,m}(\mathbf{I}) = \sqrt{\sum_{k=1}^{m}\left[\int_\Omega \|\nabla I^k\|\right]^2}.$$

This PDE comes from a minimisation process, which use coupling between vector components in the functional expression.
But, if we introduce the ξ^i direction ($\xi^i \perp \eta^i = \frac{\nabla I^i}{\|\nabla I^i\|}$), then

$$\frac{\partial I^i}{\partial t} = \alpha\ (I_0^i - I^i) + \frac{A^i}{\|\nabla I^i\|}\ I_{\xi^i\xi^i} \qquad \left(A^i = \frac{TV_{n,1}(I^i)}{TV_{n,m}(\mathbf{I})}\right)$$

The only coupling terms in the final PDE is A^i, which weight the diffusion intensity in each image channel. The diffusion is uni-directional and the smoothing direction is independent for each channel, which leads to the problem of decoupled diffusion (Fig.1).

Despite the uni-directional diffusion, texture effects are less visible in flat regions, because each channel diffuses in a different direction ξ^i. For color images, it corresponds to a *color blending effect*. Anyway this advantage becomes a drawback in contour regions : A *local vector geometry* is not taken into account, and edges evolve individually in different directions, component by component. Edges tend to be smoothed.

3.3. A geometric diffusion PDE approach

Here is our approach, considered as an extension of our previous work [34, 20, 19, 18]. It is based on a geometric viewpoint of the diffusion process.
The vector gradient norm $N(\mathbf{I}) = \sqrt{\lambda_+}$ is a local geometry indicator :

- $N(\mathbf{I})_{(M)} \approx 0$: The point M is in a flat region.
- $N(\mathbf{I})_{(M)} \gg 0$: The point M is on an edge.

Following the behaviour of Φ-function diffusions, we want an isotropic smoothing when $N(\mathbf{I}) \simeq 0$ and a tangent smoothing along the vector edge elsewhere (in the ξ direction, coming from the Di Zenzo equations). Then, a natural extension of Φ-functions diffusion for the vector case is :

$$\frac{\partial \mathbf{I}}{\partial t} = \frac{\Phi^{'}(\lambda_+)}{\lambda_+}\ \mathbf{I}_{\xi\xi} + \Phi^{''}(\lambda_+)\ \mathbf{I}_{\eta\eta} + \alpha\ (\mathbf{I}_0 - \mathbf{I}) \tag{4}$$

where $\Phi()$ is one of the Φ-function used for the classic scalar case. Note that this PDE doesn't come from a variational formulation, and diffusion coefficients can be chosen "by hand", depending on the smoothing behaviour we desire. For instance, the following equation always diffuses the image, even on high gradients areas :

$$\frac{\partial \mathbf{I}}{\partial t} = g_\tau(\sqrt{\lambda_+})\,\mathbf{I}_{\eta\eta} + \mathbf{I}_{\xi\xi} + \alpha\,(\mathbf{I}_0 - \mathbf{I})$$

where $g_\tau(.) : \mathbb{R} \to \mathbb{R}$ is a decreasing function

$$g_\tau(s) = \exp\left(-\frac{s^2}{2\,\tau^2}\right). \tag{5}$$

In this case, τ is a fixed parameter and represents the threshold between anisotropic and isotropic smoothing.
The diffusion behaviour of these PDEs is :

- In homogeneous areas ($g \approx 1$), the noise is removed efficiently due to a vector anisotropic diffusion which doesn't favour any smoothing direction :

$$\frac{\partial \mathbf{I}}{\partial t} \simeq \mathbf{I}_{\eta\eta} + \mathbf{I}_{\xi\xi} = \Delta\mathbf{I} = \begin{pmatrix} \Delta I^1 \\ \Delta I^2 \\ \dots \\ \Delta I^n \end{pmatrix}$$

- Along the edges ($g \to 0$), the diffusion is parallel to the *vector contour* ξ :

$$\frac{\partial \mathbf{I}}{\partial t} \simeq c_{\xi\xi}\,\mathbf{I}_{\xi\xi} = c_{\xi\xi}\begin{pmatrix} I^1_{\xi\xi} \\ I^2_{\xi\xi} \\ \dots \\ I^n_{\xi\xi} \end{pmatrix}$$

  There is noise elimination *and* vector contour conservation.

⇒ The coupling is strongly used in order to analyse a *local vector geometry* of the image, and so perform a coherent smoothing process.

- Weighted data attachment term avoid the solution being too different from the initial image. The result at convergence is not over-smoothed.

3.4. Comparisons on a synthetic color image :

We tested the described methods on a very noisy color synthetic image (fig.4). It shows the different behaviours of the diffusion equations.

- Pure isotropic PDE clears the noise very well, but also edges (fig.4c).
- Sapiro-Ringach PDE eq.(2) introduces some texture effects in flat regions (fig.4d)
- TVnm equation eq.(3) suffers of color blending (particularly near the edges).
- Our diffusion PDE eq.(4) clears the noise very well in homogeneous areas, while preserving color edges.

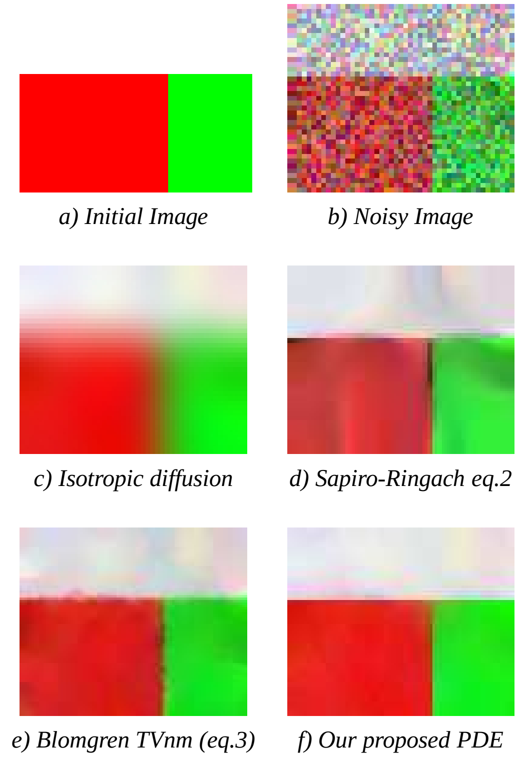


Fig. 4. Comparison on a synthetic color image

4. REDUCING THE BLUR EFFECT

Reducing the blurred edges is a part of the image restoration process. In this section, we propose to extend the scalar shock filters method [23] to the vector case, using the geometric view of vector fields. Then, we couple shock filters and vector diffusion in a single *vector image restoration* equation.

4.1. Shock filters in vector case

Scalar shock filters allow to enhance blurred edges without any knowledge of the convolution mask. It consists in raising the edges in the gradient direction ∇I : (Osher and Rudin [23]) :

$$\frac{\partial I}{\partial t} = -\mathrm{sign}(I_{\eta\eta})\,\|\nabla I\|$$

which has the following effect on the image (Fig.5 represents a slice of the local image, in the ∇I direction).

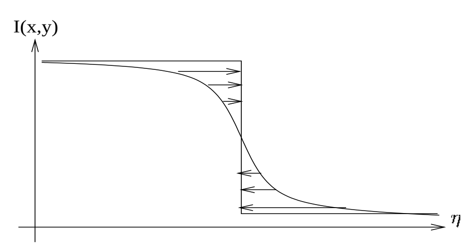


Fig. 5. Principle of scalar shock filter

For vector images, we want to raise each vector component of $\mathbf{I}$ in the same direction η of the vector geometry. We also add a weighting term that adapts the intensity of the shock filter process in order to enhance only edges and not flat regions (We used the function g_τ already defined in eq.(5)) :

$$\frac{\partial \mathbf{I}}{\partial t} = -\left(1 - g_\tau(\sqrt{\lambda_+})\right)\ \mathbf{U} \qquad (6)$$

where

$$\mathbf{U} = \begin{pmatrix} \text{sign}(I^1_{\eta\eta})\|I^1_\eta\| \\ \text{sign}(I^2_{\eta\eta})\|I^2_\eta\| \\ \ldots \\ \text{sign}(I^n_{\eta\eta})\|I^n_\eta\| \end{pmatrix}$$

Here, τ is a threshold that decides if the current point is on an edge or in a homogeneous area.

Fig.6 shows the application of this vector shock filter on a blurred color image.

a) Blurred image *b) Enhanced image*

Fig. 6. Color shock filter application

4.2. Coupling anisotropic diffusion and shock filters

We propose to combine the diffusion term eq.(4) with the shock filter term eq.(6) in a single *vector image restoration PDE* :

$$\begin{array}{rl} \frac{\partial \mathbf{I}}{\partial t} = & \frac{\Phi^{'}(\lambda_+)}{\lambda_+}\, \mathbf{I}_{\eta\eta} + \Phi^{''}(\lambda_+)\, \mathbf{I}_{\xi\xi} \\ + & \alpha_a\, (\mathbf{I} - \mathbf{I}_0) \\ - & \alpha_c\, (1 - g_\tau(\sqrt{\lambda_+}))\ \mathbf{U} \end{array} \qquad (7)$$

Free parameters α_a and α_c weight the importance of the shock filters and the data attachment towards the diffusion process.

5. NORM CONSTRAINED RESTORATION

Recently, some authors have proposed to smooth vector field with a constrained norm [25, 8, 11, 5]. It consists of evolving a PDE on a field where each vector keeps its initial norm. Here we propose a geometric viewpoint of the problem, allowing to extend norm constrained evolutions to all kind of PDEs, in particular shock filters. Then we use our equation (7), in order to build a *norm constrained restoration PDE for vector images*.

5.1. A geometric formulation

The vector norm must be preserved during the PDE evolution :

$$\forall M \in \Omega, \forall t, \quad \|\mathbf{I}(M)\|^2 = cste$$

Derivating this equation with respect to t gives an equivalent expression :

$$\forall M \in \Omega, \forall t, \quad 2\,\mathbf{I}(M)\,.\,\frac{\partial \mathbf{I}(M)}{\partial t} = 0$$

It means that the PDE *velocity vector* $\frac{\partial \mathbf{I}(M)}{\partial t}$ must be orthogonal to the vector $\mathbf{I}(M)$, in order to preserve its norm. Suppose then we have a vector PDE of the general form :

$$\frac{\partial \mathbf{I}}{\partial t} = \boldsymbol{\beta} \qquad \text{where } \boldsymbol{\beta} \in \mathbb{R}^n$$

Adding the norm constraint can be naturally done by *projecting the velocity* $\boldsymbol{\beta}$ to the hyperplane, orthogonal to $\mathbf{I}$, which is :

$$\mathcal{P}^\perp_{\mathbf{I}}(\boldsymbol{\beta}) = \boldsymbol{\beta} - \left(\frac{\boldsymbol{\beta}\,.\,\mathbf{I}}{\|\mathbf{I}\|^2}\right)\ \mathbf{I}$$

Then, the following equation ensures that the norm of $\mathbf{I}$ is preserved during the evolution :

$$\frac{\partial \mathbf{I}}{\partial t} = \boldsymbol{\beta} - \left(\frac{\boldsymbol{\beta}\,.\,\mathbf{I}}{\|\mathbf{I}\|^2}\right)\ \mathbf{I} \qquad (8)$$

The geometric explanation is simple (Fig.7).

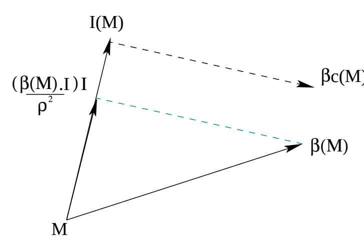


Fig. 7. Geometric view of the norm constraint

During the PDE evolution, $\mathbf{I}(M)$ does a *pure rotation* and preserves its norm. Then, for the particular case of vector field restorations under constrained norm, we can choose $\boldsymbol{\beta}$ to be the expression of eq.(7). It allows the use of shock filters as well as accurate diffusion for norm constrained vector fields, and extends naturally previous works on this subject [25, 8, 11, 5].

5.2. Applications of constrained norm PDEs

We used this equation in order to restore noisy chromaticity color images and normalised 2D optical flow directions. The restoration of chromaticity have been studied in [8, 11]. It consists of denoising only chromaticity data of a color image. Indeed, each vector $\mathbf{I}(M) = (R, G, B)$ of a color image can be decomposed as :

$$\mathbf{u}(M) = \frac{\mathbf{I}(M)}{\|\mathbf{I}(M)\|} \qquad : \textit{the chromaticity vector}$$

and

$$l = \|\mathbf{I}(M)\| \qquad : \text{the } \textit{brightness}.$$

$\mathbf{u}$ is normalised and contains only color information of the pixel. If we know that the noise is only present on $\mathbf{u}$, we can use the eq.(8) on $\mathbf{I}$, in order to restore $\mathbf{u}$ and let the intensity l of each pixel unchanged.

6. EXPERIMENTAL RESULTS

We used our vector restoration PDE (7) and our norm constrained equation (8) for different applications :

- Restoration of normalised 2D vector fields, representing direction flows ($\mathbb{R}^2$ vectors) : Fig.9 shows the importance of norm constraint on a synthetic image. Note how the vector norms are smoothed with unconstrained PDE (Fig.9.c).

- Restoration of color images in the (R, G, B) space (dealing with $\mathbb{R}^3$ vectors) with RGB-noise and chromaticity noise. The knowledge of the chromaticity noise model allows to restore very well the initial noisy color image.

Simple explicit numerical schemes with finite differences were used. As usual, shock filter process requires special schemes, due to its hyperbolic nature : classic *minmod schemes* work well (see [23, 17] for more details). Other color results are visible in the author web pages :
http://www-sop.inria.fr/robotvis/personnel/David.Tschumperle and
http://www-sop.inria.fr/robotvis/personnel/der

7. CONCLUSION

We have proposed a new vector PDE that restores constrained and unconstrained vector field of any dimension. Applications are multiple : color image restoration, optical flow regularisation, multiscale representation... The equation we proposed doesn't come from an energy minimisation, but was built in order to follow a number of desired geometric diffusion properties, which ensure a very efficient restoration process. The main key is the use of a *local vector geometry* to adapt the behaviour of the diffusion and shock filters, using the coupling between vector components as good as possible. The general PDE for norm constrained evolution (eq.8) can also be the start for other well know vector problems (optical flow computation, orientation analysis, color image interpolation, ...). We are working on these interesting subjects.

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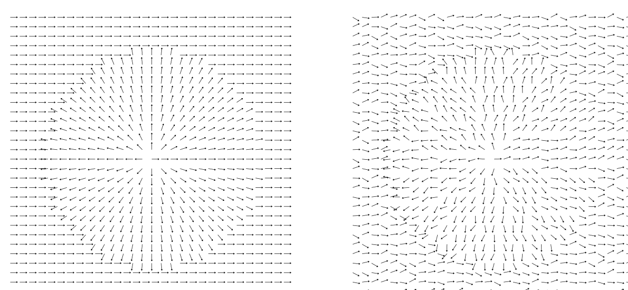
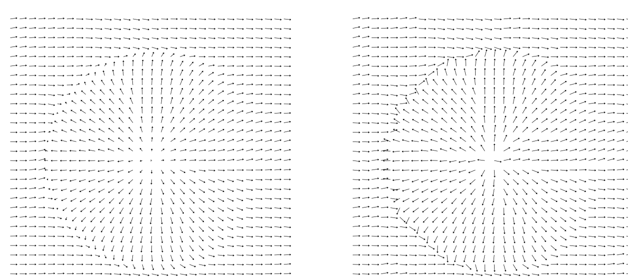
a) With R,G,B Gaussian noise. b) Our color restoration (eq.7).

Fig. 8. Unconstrained Color restoration

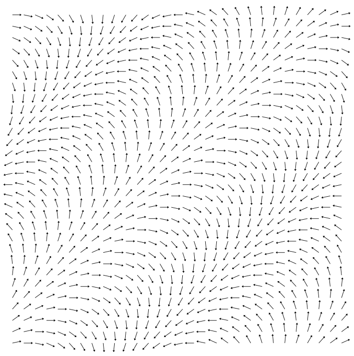


a) Synthetic direction field b) With angle noise ($\sigma = 30^o$)

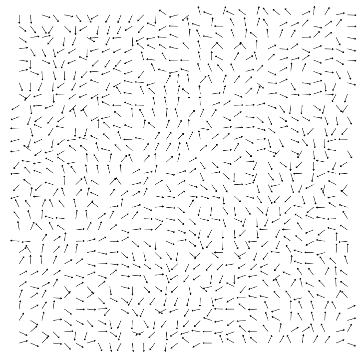


c) With unconstrained PDE(eq.7) d) With constrained PDE (eq.8)

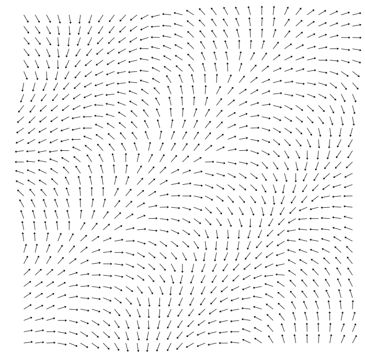
Fig. 9. Restoration of direction field



a) *Normalized "Sinus" flow*



b) *With direction noise*

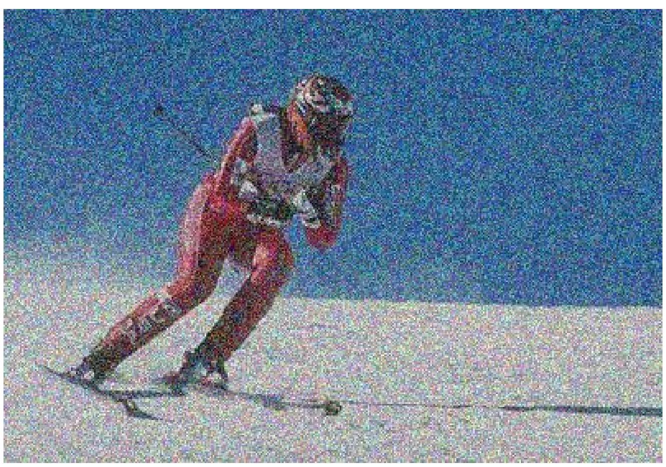


c) *Restored flow (eq.8)*

a) *Noisy chromaticity image*

b) *With unconstrained PDE (7)*

c) *With constrained PDE (8)*



a) *Noisy color image (gaussian on RGB)*

b) *Restoration with Eq.2*

c) *Restoration with TVnm Eq.3*

d) *Our color restoration (eq.7)*