Diffusion PDE's on Vector-valued Images : Local Approach and Geometric Viewpoint

D. Tschumperlé R. Deriche

I.N.R.I.A, Projet ROBOTVIS/ODYSSEE, 2004 Rte des Lucioles, BP 93, 06902 Sophia Antipolis, France

Abstract

[1] *We study multivalued diffusion PDE's (Partial Differential Equations) and their application to color image processing. The analysis of classic scalar diffusion PDE's leads to a new multivalued regularization equation which is coherent with a local vector image geometry. Then, we are interested in constrained regularization problems, where vector norm constraints have to be considered. A general extension for unit vector regularization is then proposed. Finally, experimental results of color image restoration are presented.*

Introduction

For many years, the restoration of noisy and blurred digital data has been widely studied, and many algorithms based on variational or stochastic formulations have tried to solve this ill-posed problem. The variational methods based on *diffusion PDE's* (Partial Differential Equations) have particularly proven their efficiencies in order to regularize images while preserving important data discontinuities, that often contain edge informations. Actually, one of the first steps was initiated more than ten years ago, with the pionnering work of Perona & Malik [34], who proposed an anisotropic diffusion PDE reconstruction algorithm to smooth grey-valued images while preserving the edges, overcoming the restrictions imposed by linear filtering techniques. Since then, many authors have proposed and studied well-posedness PDE's that tackle the problem of scalar image regularization. We can cite for instance papers from Alvarez [2, 1], Aubert [10], Chambolle & Lions [7], Chan [5], Cohen [13], Cottet & Germain [14], Hamza & Krim [18], Kornprobst & Deriche [22, 23, 24], Malladi & Sethian [27], Mumford & Shah [29, 46], Morel [28], Nordström [30], Osher & Rudin [39], Perona & Malik [34], Polyak [35], Proesman [38], Sapiro [6, 43, 44, 45], Weickert [58, 60, 61] and You [64]. More recently and thanks to increased computer memory capacity, the problem of regularizing images of *vector-valued features* has become an active research area, because of the large number of possible applications, including various computer vision tasks : color image restoration, [5, 20, 45, 48, 53, 58], regularization of optical flows and direction fields [8, 21, 33, 50, 49, 55], image inpainting [4, 9], regularization of diffusion MRI, [15, 12, 36, 54, 57], scale space analysis [1, 62]. Proposed algorithms rely on various frameworks, such as image variation minimization [3, 5, 46], local geometry-based diffusion [45, 53], flow of manifolds embedded in higher dimensional space minimizing Polyakov actions [47] or diffusion of chemical concentrations [58].

In this paper, we are first interested in the geometric properties that should satisfy an efficient image restoration PDE, illustrated with the well known ϕ-function framework for scalar images. Then we focus on vector-valued data (and particularly color images) and compare different vector geometry definitions, based on the Di Zenzo calculus [65]. It allows us to analyze reference works in the domain of vector image diffusion with PDE's [5, 45, 47, 58] using *a local approach and a geometric viewpoint.* From this study a natural generalization of the ϕ-function based diffusion to multi-valued images will emerge. Within the same idea, we propose to enhance blurred multi-valued edges, thanks to a vector extension of the *shock filter method* [31]. Proposed equations are then derived in order to process fields of *norm constrained vectors.* Finally, we illustrate the different applications of our vector regularization PDE's, including restoration of color images, optical flows, direction fields and chromaticity noise removal.

1 Anisotropic diffusion and local geometry of images

Few years ago, the ϕ-function formulation enabled the unification of many proposed anisotropic regularization

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PDE's acting on scalar images within a common variational framework [10, 23, 11]. Let Ω be a spatial 2D domain (for instance, $\Omega = [a, b]$). We assume Neumann boundary conditions on $\partial\Omega$. A noisy *scalar image* $I_0 : \Omega \to \mathbb{R}$ can be regularized by minimizing the following ϕ-functional

$$E(I) = \int_\Omega \left[\frac{\alpha}{2} \, (I - I_0)^2 + \phi(\|\nabla I\|) \right] \, d\Omega$$

The fixed parameter $\alpha > 0$ prevents the expected solution from being too different from the given noisy image I_0, while $\phi : \mathbb{R} \to \mathbb{R}$ is *an increasing function*, which controls the regularization behaviour. The minimization is often performed via the corresponding *diffusion PDE evolution*, coming from the Euler-Lagrange equations of $E(I)$:

$$\frac{\partial I}{\partial t} = \alpha(I_0 - I) + \text{div}\left(\frac{\phi^{'}(\|\nabla I\|)}{\|\nabla I\|} \, \nabla I \right) \qquad (1)$$

A lot of proposed scalar regularization methods can then be expressed by finding the corresponding ϕ-function : Minimal surfaces [10], Geman & McClure [17], Perona & Malik [34], Total variation [40], Tikhonov [51] among other examples.
Moreover, an interesting development of (1) has been proposed in [23] :

$$\frac{\partial I}{\partial t} = \alpha(I_0 - I) + \phi^{''}(\|\nabla I\|) \, I_{\eta\eta} + \frac{\phi^{'}(\|\nabla I\|)}{\|\nabla I\|} \, I_{\xi\xi} \quad (2)$$

where $I_{\eta\eta} = \eta^T H \eta$ and $I_{\xi\xi} = \xi^T H \xi$ are the second spatial derivatives of I in the directions of the gradient $\eta = \nabla I / \|\nabla I\|$ and its orthogonal $\xi = \eta^\perp$ (here H denotes the *hessian* of I and $I_{\eta\eta}$ and $I_{\xi\xi}$ are then *directional 1D second derivatives*). Note that if $M \in \Omega$ is located on an image discontinuity (edge), $\xi_{(M)}$ is a vector tangent to this edge (since it is tangent to the isophotes). Geometrically speaking, the PDE (2) performs then *two sequential and directional 1D diffusions* that smooth the image data in the isophote direction ξ with a weight c_ξ and in its orthogonal direction η with a weight c_η :

$$\frac{\partial I}{\partial t} = \alpha(I_0 - I) + c_\eta \, I_{\eta\eta} + c_\xi \, I_{\xi\xi} \qquad (3)$$

with

$$c_\xi = \phi^{'}(\|\nabla I\|) / \|\nabla I\| \quad \text{and} \quad c_\eta = \phi^{''}(\|\nabla I\|) \qquad (4)$$

Different choices of ϕ obviously lead to different local diffusion behaviours. Anyway for image restoration purposes, there are natural *geometric properties* that should be verified by the coefficients c_ξ and c_η :
- The functions c_ξ and c_η must be *positive*, in order to avoid instable inverse diffusion along η or ξ.
- When the local geometry is flat and doesn't contain any edges ($\|\nabla I\| \to 0$), *the diffusion should be isotropic* i.e with no prefered diffusion directions since η and ξ do not represent signifiant orientations in this case :

$$c_\eta \simeq c_\xi = \beta > 0 \;\; \text{then} \;\; \frac{\partial I}{\partial t} \simeq \beta \, (I_{\xi\xi} + I_{\eta\eta}) = \beta \, \Delta I$$

-On high gradient regions ($\|\nabla I\| \gg 0$), the current point may be located on an edge, the diffusion should be done *only along the edge direction* ξ, in order to preserve it (anisotropic diffusion) :

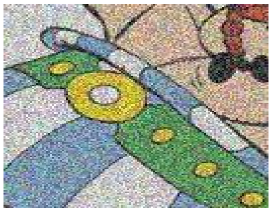
$$c_\xi \gg c_\eta \quad \text{and} \quad c_\eta \simeq 0 \qquad \text{then} \qquad \frac{\partial I}{\partial t} \simeq c_\xi \, I_{\xi\xi}$$

For instance, the hypersurface function $\phi(s) = 2\sqrt{1+s^2} - 2$ defined in [10] satisfies these properties. In [23], the authors proposed also to fix the smoothing intensities $c_\eta = g_\tau(\|\nabla I\|)$ (decreasing function) and $c_\xi = 1$, which ensures a permanent noise removal, but tends to smooth sharp corners, corresponding to very high gradients points. Note that in this case, no function ϕ can be found to have the same geometric behaviour, and the corresponding PDE cannot be interpreted as a gradient descent of an energy functional anymore. Nevertheless, this means that one can choose c_ξ and c_η to fit more precise geometric constraints that those verified by the original ϕ function formulation.

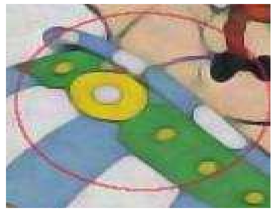
The observation of scalar image diffusion equations teaches us that a regularization equation should adapt its diffusion behaviour to the *local geometry of the image*, defined by *edge indicators* and *edge orientations*. For scalar images, such attributes are respectively given by $\|\nabla I\|$ and by the orientation basis $(\ \eta\ ,\ \xi\ )$. Regularization processes acting on *vector-valued images* $\mathbf{I}$ need to be driven by equivalent geometric attributes, taking the coupling between vector channels I_i into account (computing such a geometry is the matter of section 2).
Using separate scalar PDE's on each component I_i of a multi-valued image $\mathbf{I}$ is then useless : each channel are diffusing with *different local geometries* $\|\nabla I_i\|$ and $(\ \eta_i\ ,\ \xi_i\ )$. The resulting image is blended, and vector edges are falsely smoothed, as illustrated on Figure 1.

2 Defining a vector geometry

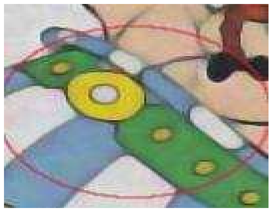
We concentrate now our attention on *vector images* $\mathbf{I}(M) : \Omega \to \mathbb{R}^n$ ($n = 3$ for color images). We denote



a) Noisy image



b) Channel by channel



c) Vector approach

Figure 1: *Channel by channel approach vs vector-based PDE, applied on a noisy color image (considering the (R,G,B) vector space).*

by I_i the i^{th} image channel $(1 \leq i \leq n)$:

$$\forall M \in \Omega, \quad \mathbf{I}(M) = (\ I_1(M)\ ,\ I_2(M)\ ,\ \ldots\ ,\ I_n(M)\ )$$

The idea is to find a *local vector geometry* on each point $M \in \Omega$, defined by :
- *A vector gradient norm* $\mathcal{N}_{(M)}$ that detect edges and corners when its value becomes high. $\mathcal{N}$ should then reduce to $\|\nabla I\|$ for scalar images $(n = 1)$.
- Two corresponding *variation orientations* $\theta_{+_{(M)}}$ and $\theta_{-_{(M)}}$ that are respectively orthogonal and tangent to the vector edges, if there are any.

One approach would be to compute first a scalar image $f(\mathbf{I})$, using a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ that models the human perception of vector edges. For instance, one could choose the luminance function $f = L^*$ for color images. Then we could define $\theta_+ = \nabla f(\mathbf{I})/\|\nabla f(\mathbf{I})\|$, and $\mathcal{N} = \|\nabla f(\mathbf{I})\|$. However, there are mathematically no functions f that can detect all possible vector variations : For instance, the luminance function wouldn't be able to detect iso-luminance contours.

Another solution has been proposed by Di Zenzo in [65]. He considers a multi-valued image $\mathbf{I}$ as a $2D \rightarrow$ n-D vector field, and looks for the local variations of the norm $\|d\mathbf{I}\|^2$, mainly given by a variation matrix $\mathbf{G} = (g_{i,j})$. If we denote by $\mathbf{X} = (x,y)^T$, we get :

$$\|d\mathbf{I}\|^2 = d\mathbf{X}^T\ \mathbf{G}\ d\mathbf{X} \quad \text{where} \quad \mathbf{G} = \sum_{i=1}^{n} \nabla I_i\ \nabla I_i^T$$

For color images $\mathbf{I} = (R,G,B)$ the symmetric and semi-positive matrix $\mathbf{G}$ is then :

$$\mathbf{G} = \begin{pmatrix} R_x^2 + G_x^2 + B_x^2 & R_x R_y + G_x G_y + B_x B_y \\ R_x R_y + G_x G_y + B_x B_y & R_y^2 + G_y^2 + B_y^2 \end{pmatrix} \tag{5}$$

The positive eigenvalues $\lambda_{+/-}$ of $\mathbf{G}$ are the maximum and the minimum of $\|d\mathbf{I}\|^2$ and the orthogonal eigenvectors θ_+ and θ_- are the corresponding variation orientations :

$$\lambda_{+/-} = \frac{g_{11} + g_{22} \pm \sqrt{\Delta}}{2} \tag{6}$$

and

$$\theta_{+/-}\ /\!/\ \begin{pmatrix} 2\ g_{12} \\ g_{22} - g_{11} \pm \sqrt{\Delta} \end{pmatrix}$$

where $\Delta = (g_{11} - g_{22})^2 + 4\ g_{12}^2$. Note then that $\lambda_+ \geq \lambda_- \geq 0$.

The local *orientations of the vector edges* are then naturally defined by the orthogonal bases $(\theta_+\ ,\ \theta_-)$. Concerning the $\lambda_{+/-}$, three geometric cases could be considered (an example of color image illustrates these cases, Figure 2a).
- If $\lambda_+ \simeq \lambda_- \simeq 0$, there are very few vector variations around the current point : the region is *flat* and doesn't contain any edges or corners (look at the inside of the strips in Figure 2a).
- If $\lambda_+ \gg \lambda_-$, there are a lot of vector variations. The current point may then be located on a *vector edge* (the edges of the strips in Figure 2a).
- If $\lambda_+ \simeq \lambda_- \gg 0$, we are located on a saddle point of the image, which can possibly be a *vector corner* (the intersections of the strips in Figure 2a).

Three differents choices of vector gradient norms $\mathcal{N}$ can then be made :
- $\mathcal{N} = \sqrt{\lambda_+}$, as a natural extension of the scalar gradient norm viewed as the value of maximum variations [41, 42, 52, 53] (Figure 2b).
- $\mathcal{N}_- = \sqrt{\lambda_+ - \lambda_-}$, also called *coherence norm*, have been choosen in [45, 57, 58] to measure vector variations. Note that this norm fails to detect discontinuities that are saddle points of the image (Figure 2c).
- $\mathcal{N}_+ = \sqrt{\lambda_+ + \lambda_-}$, also denoted by $\|\nabla\mathbf{I}\|$ is often choosen [49, 3, 5, 32, 46, 55, 54] since it detects edges and corners in a good way, and it is easy to compute :

$$\mathcal{N}_+ = \sqrt{\text{trace}(\mathbf{G})} = \sqrt{\sum_i \|\nabla I_i\|^2}$$

Note that $\mathcal{N}_+$ sometimes gives preferences to certain corners (Figure 2d), which is very interesting for image restoration purposes, since the smoothing will be attenuated on these corners (which is a desired behaviour). Note also that for the scalar case $(n = 1)$, $\mathcal{N}_+$, $\mathcal{N}_-$ and $\mathcal{N}$ naturally reduce to $\|\nabla I\|$, as in this case $\lambda_- = 0$ and $\lambda_+ = \|\nabla I\|^2$.

Once a local vector geometry is defined, we can use it as a measure in many computer vision processes acting on vector images. For instance, color edge detection

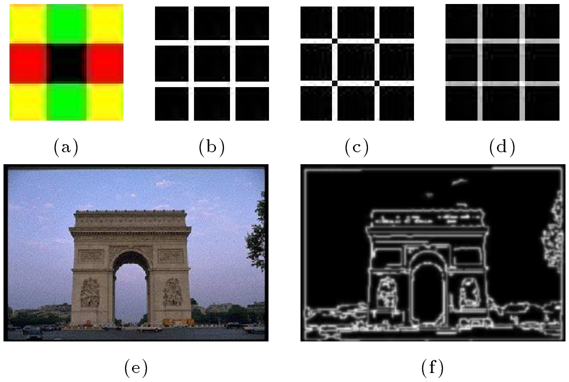


Figure 2: *Vector variations norms and application to edge detection : (a) Color image, (b)* $\mathcal{N} = \sqrt{\lambda_+}$*, (c)* $\mathcal{N}_- = \sqrt{\lambda_+ - \lambda_-}$*, (d)* $\mathcal{N}_+ = \sqrt{\lambda_+ + \lambda_-}$*, (e) Original color image, (f) Edge detection with the* $\mathcal{N}$ *norm*

can be performed by finding the local maxima of $\mathcal{N}$ in the θ_+ direction (Figure 2e,f and [25]). This vector geometry computation has also been integrated for color image segmentation purposes in [41, 42].

3 Vector diffusion PDE's

We will now analyze the recent proposed diffusion PDE's acting on vector images with respect to their local geometric behaviour. We will use the previous notations ξ, η to refer to the scalar local geometry (section 1), and θ_+, θ_-, $\mathcal{N}_+$, $\mathcal{N}_-$, $\mathcal{N}$ and the matrix $\mathbf{G}$ to refer to the vector local geometry (section 2). This Section concludes in a comparative figure Figure 3 that illustrates the behaviour of each proposed equation applied on a higly noisy synthetic color image.

3.1 Color Total Variation

In order to regularize vector-valued images, Blomgren and Chan in [5] proposed to minimize a measure of a *color total variation* TV_n (which reduces to the scalar TV when $n = 1$) :

$$\min_{\mathbf{I}} TV_n(\mathbf{I}) = \sqrt{\sum_{i=1}^{n} \left[\int_\Omega \|\nabla I_i\| \, d\Omega \right]^2}.$$

Minimizing the TV_n leads to the following vector diffusion PDE (written in a component by component style) :

$$\frac{\partial I_i}{\partial t} = \frac{\int_\Omega \|\nabla I_i\|}{TV_n(\mathbf{I})} \, div\left(\frac{\nabla I_i}{\|\nabla I_i\|}\right) \tag{7}$$

Note that introducing the orthogonal gradient direction $\xi_i = (\nabla I_i / \|\nabla I_i\|)^\perp$, defined in section 1, this PDE writes :

$$\frac{\partial I_i}{\partial t} = A_i \, \frac{I_{\xi_i \xi_i}}{\|\nabla I_i\|} \qquad \text{where} \qquad A_i = \frac{\left(\int_\Omega \|\nabla I_i\| \, d\Omega\right)}{TV_n(\mathbf{I})}$$

The diffusion is then a *channel by channel TV*, weighted by a coupling term A_i which is *constant for a whole channel* I_i. No local vector interactions are used : Noisy vector edges that not clearly appear in each channel of the image, may be smoothed by this method (look at the bottom of the central grey object in Figure 3). Otherwise, this method is well adapted for removing uncorrelated noise. Minimizing a vector coupled functional do not necessary lead to a PDE that implicitly considers a local vector geometry.

3.2 Coherence Enhancing Diffusion

In [58, 59, 61], Weickert viewed the image regularization process as the diffusion of chemical concentrations and propose to apply this diffusion PDE, inspired from the field of fluid physics :

$$\frac{\partial I_i}{\partial t} = \mathrm{div}\left(\mathbf{D} \, \nabla I_i\right) \tag{8}$$

(Note that this PDE may not come from a variational principle)
$\mathbf{D} = \lambda_1 \mathbf{u}\mathbf{u}^T + \lambda_2 \, \mathbf{v}\mathbf{v}^T$ is the *diffusion tensor* (i.e a symmetric and positive definite matrix) that possesses λ_1,λ_2 as positive eigenvalues and $\mathbf{u}$,$\mathbf{v}$ as corresponding orthonormal eigenvectors and that drive the regularization process : the amount of diffusion in the directions $\mathbf{u}$ and $\mathbf{v}$ will be weighted by λ_1 and λ_2.

For the particular problem of vector-valued image diffusion [58], $\mathbf{D}$ is explicitely constructed from a smoothed version $\mathbf{G}^*$ of the Di Zenzo variation matrix $\mathbf{G}$ (5), in order to possesses the following spectral elements :

$$\begin{cases} \lambda_1 = \beta \\ \lambda_2 = \beta + (1-\beta)\exp\left(\frac{-C}{(\lambda_+ - \lambda_-)^2}\right) \end{cases}$$

and

$$\begin{cases} \mathbf{u} = \theta_+ \\ \mathbf{v} = \theta_- \end{cases} \qquad (\beta, C > 0)$$

where $\theta_{+/-}$ and $\lambda_{+/-}$ are smoothed versions of the eigen elements of the matrix $\mathbf{G}^*$. This equation geometrically reads as :

- On flat regions, according to the coherence norm $\mathcal{N}_-$ (i.e $\lambda_+ \simeq \lambda_-$), the smoothing is isotropic with a weight $\beta \in [0, 1]$, since $\lambda_1 = \lambda_2 = \beta$.
- Near the edges ($\mathcal{N}_- \gg 0$ i.e $\lambda_+ \gg \lambda_-$), the diffusion

is mainly made along the vector edge orientation θ_- *but also along its orthogonal orientation* θ_+, with a weight β (note that edges may be blurred for high values of β, Figure 3d).
To avoid orthogonal smoothing in the edge regions, one can choose $\beta \rightarrow 0$, but it also suppresses the interesting isotropic smoothing behaviour in flat regions (Figure 3e).

3.3 The Beltrami Flow

With a completely different approach, Sochen and Kimmel [47, 20] found a particular case of the coherence enhancing diffusion (8). Considering a vector/color image as a 2D surface embedded in a $(n+2)$D space (this idea was also used in [63]) and minimizing a Polyakov action, they end up in the following diffusion PDE called *Beltrami Flow*, that can be expressed with the Di Zenzo matrix $\mathbf{G}$ (5), by :

$$\frac{\partial I_i}{\partial t} = \frac{1}{\sqrt{\det(\mathbf{A})}} \operatorname{div}\left(\sqrt{\det(\mathbf{A})}\ \mathbf{A}^{-1}\ \nabla I_i\right) \qquad (9)$$

where $\mathbf{A} = \mathbf{Id} + \mathbf{G}$ and $\mathbf{Id}$ is the 2×2 identity matrix. This equation is then a weighted version of eq.(8), with the following diffusion tensor :

$$D = \sqrt{\det(\mathbf{Id}+\mathbf{G})}\ (\mathbf{Id}+\mathbf{G})^{-1}$$

In this case, the spectral elements of $\mathbf{D}$ that geometrically drives the diffusion can also be written using the eigenvalues and eigenvectors of $\mathbf{G}$:

$$\lambda_1 = \sqrt{\frac{1+\lambda_-}{1+\lambda_+}} \quad , \quad \lambda_2 = \sqrt{\frac{1+\lambda_+}{1+\lambda_-}}$$

and

$$\left\{ \begin{array}{l} \mathbf{u} = \theta_+ \\ \mathbf{v} = \theta_- \end{array} \right.$$

Here, the diffusion behaviour depends mainly in how λ_+ compares to λ_-, i.e implicitely depends of the *coherence variation norm* $\mathcal{N}_-$:
- On flat regions ($\lambda_+ \simeq \lambda_-$ i.e $N_- \rightarrow 0$), the diffusion is isotropic : $\lambda_1 \simeq \lambda_2 \simeq 1$.
- Near edges, ($\lambda_+ \gg \lambda_-$ i.e $N_- \gg 0$), the diffusion is mainly done along the vector edge direction θ_-, as in this case $\lambda_1 \simeq 0$.
Note also that the weighting term of the eq.(9)

$$\frac{1}{\sqrt{\det(\mathbf{Id}+\mathbf{G})}} = \frac{1}{\sqrt{(1+\lambda_+)(1+\lambda_-)}}$$

quickly decreases the amount of diffusion near high gradients, and vector edges may be preserved for a long time during the flow (and unfortunately noisy edges too, as illustrated in Figure 3f).

3.4 Vector $\mathbf{I}_{\xi\xi}$ Diffusion

In [45], Ringach and Sapiro proposed an extension of the weighted mean curvature equation $I_t = \beta\ I_{\xi\xi}$ for the vector case. They naturally used the Di Zenzo vector geometry to design this regularization PDE :

$$\frac{\partial \mathbf{I}}{\partial t} = g(\lambda_+ - \lambda_-)\ \mathbf{I}_{\theta_-\theta_-} \qquad (10)$$

where $g : \mathbb{R} \rightarrow \mathbb{R}$ is a positive decreasing function, avoiding the smoothing of regions containing large gradients. It was one of the first attempts to construct a vector diffusion PDE directly from a local geometry viewpoint. At a given point, all channels I_i evolve in the direction of vector edges and with a mutual intensity. Anyway, some drawbacks subsist :

- The coherence norm $\mathcal{N}_- = \sqrt{\lambda_+ - \lambda_-}$ may not detect some vector corners (Figure 2d).
- In flat regions ($\mathcal{N}_- \rightarrow 0$), the diffusion is made along a direction θ_-, which is mainly directed by the noise. Texture effects may result from this uni-directionnal smoothing in homogeneous areas (look at the yellow circle in Figure 3g).

3.5 A new geometric designed diffusion PDE

Our approach is based on some regularization heuristics that would be used if one had to restore a vector-valued image (especially a color image). We use the vector gradient norm $\mathcal{N}_+ = \sqrt{\lambda_+ + \lambda_-}$ to detect the local configuration of the image (flat region or edges), for reasons explained in section 2.
- On almost constant color (or vector) regions, a natural idea would be to smooth isotropically the region, in order to remove the noise. The diffusion equation must then be close to :

$$\frac{\partial I}{\partial t} = \Delta I \qquad \text{when} \qquad \mathcal{N}_+ \rightarrow 0$$

- On edges and corners, we want to smooth the image with less intensity and in the direction of the *vector edge*, which means that :

$$\mathcal{N}_+ \gg 0 \qquad \Longrightarrow \qquad \frac{\partial I}{\partial t} = \beta(\mathcal{N}_+)\ \mathbf{I}_{\theta_-\theta_-}$$

(β is a decreasing function).
As described in Section 1, these geometric properties are naturally verified by diffusion equations of the form :

$$\frac{\partial \mathbf{I}}{\partial t} = c_{\theta_-}(\mathcal{N}_+)\ \mathbf{I}_{\theta_-\theta_-} + c_{\theta_+}(\mathcal{N}_+)\ \mathbf{I}_{\theta_+\theta_+} + \alpha\ (\mathbf{I}_0 - \mathbf{I}) \qquad (11)$$

where $c_{\theta_-} : \mathbb{R} \rightarrow \mathbb{R}$ and $c_{\theta_+} : \mathbb{R} \rightarrow \mathbb{R}$ are decreasing functions, like those proposed in the scalar case.
Note that with this geometric approach, we can obtain the original diffusion behaviours of the ϕ-functions, if we choose c_{θ_-} and c_{θ_+} to be defined by (4). This equation is designed to fully adapt its smoothing behaviour to the local vector geometry of the image and so performs a coherent restoration process (Figure 3h).

3.6 Comparisons of the PDE's on a synthetic color image

We tested the described methods eq.(7),(8),(9),(10),(11) on a very noisy color synthetic image (Figure 3).

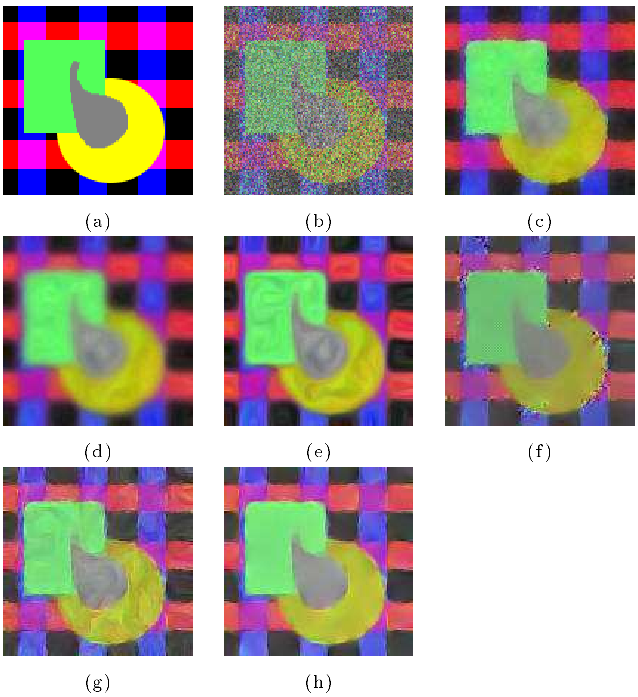


Figure 3: *Comparison of vector diffusion PDE's on a synthetic color image : (a) Color image, (b) Noisy image, (c) Color TV eq.(7), (d) Coherence Enhancing eq.(8) ($\beta = 0.05$), (e) Coherence Enhancing eq.(8) ($\beta = 0.01$), (f) Beltrami flow eq.(9), (g) Vector $\mathbf{I}_{\xi\xi}$ eq.(10), (h) Geometric Vector PDE eq.(11)*

A data attachment term $\alpha(\mathbf{I}_0 - \mathbf{I})$ (with $\alpha = 0.01$) has been added to all equations, and the PDE flows have been applied on the (R, G, B) color space. Figure 3 shows the results at convergence. The added noise is highly correlated between the image channels and has been obtained in noisying the (H, S, V) color space of the original synthetic image with uniform noise. It avoid to favour PDE's working separatly on the vector components (noise in real vector images are seldom uncorrelated) This figure allows to experimentally analyze the expected local diffusion behaviour of each vector diffusion PDE's, on flat regions and around the edges and corners.

4 Reducing the blur effect

Reducing the blurred edges can also be a part of an image restoration process. The scalar shock filter method proposed in [31] enhances blurred edges in grey-valued images without any knowledge of the convolution mask. It operates by raising the signal in the gradient direction η (Figure 4) :

$$\frac{\partial I}{\partial t} = -\text{sign}(I_{\eta\eta}) \, \|\nabla I\|$$

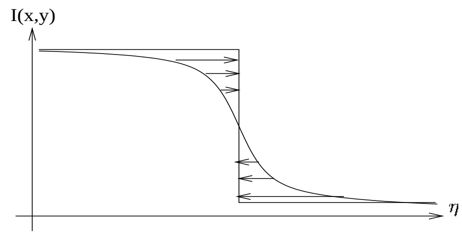


Figure 4: *Principle of shock filters*

For vector images, we naturally would like to raise each vector component I_i of $\mathbf{I}$ in the same direction θ_+ of the vector discontinuities. We also add a weighting term that adapts the intensity of the shock filter process in order to enhance only edges and not flat regions :

$$\frac{\partial \mathbf{I}}{\partial t} = -\left(1 - g(\mathcal{N}_+)\right) \, \mathbf{U} \qquad \left(\text{ with } \mathcal{N}_+ = \sqrt{\lambda_+ + \lambda_-} \;\right) \tag{12}$$

where $g : \mathbb{R} \rightarrow [0, 1]$ is a decreasing function and $\mathbf{U}$ is the *shock filter vector* whose components are :

$$U_i = \text{sign}\left(\frac{\partial^2 I_i}{\partial \theta_+^2}\right) \left\| \frac{\partial I_i}{\partial \theta_+} \right\|$$

Some results of applications can be found in the result Section 6. We also propose to add this vector shock filter term to our geometric ϕ-function diffusion PDE eq.(11), to obtain a single *vector image restoration PDE*, allowing to clear local noise *and* enhance blurred edges :

$$\frac{\partial \mathbf{I}}{\partial t} = c_{\theta_-} \, \mathbf{I}_{\theta_-\theta_-} + c_{\theta_+} \, \mathbf{I}_{\theta_+\theta_+} + \alpha \, (\mathbf{I}_0 - \mathbf{I}) - \beta(\mathcal{N}_+) \, \mathbf{U} \tag{13}$$

where $\beta : \mathbb{R} \rightarrow \mathbb{R}$ is an increasing function. The positive parameters α and β weight the importance of the shock filters and the data attachment towards the diffusion process. An example of color image restoration using eq.(13) is illustrated in Figure 6f.

5 Norm constrained restoration

Recently, some authors have proposed smoothing vector fields with equations that preserve the vector norm [33, 49, 8, 3]. The obtained PDE's are constrained versions of classic vector diffusion equations. Here is a geometric viewpoint of this problem. Indeed, the norm constraint is equivalent to :

$$\|\mathbf{I}(x,y)\|^2 = \text{constant} \Longleftrightarrow 2\,\mathbf{I}(M)\,.\,\frac{\partial \mathbf{I}(x,y)}{\partial t} = 0$$

This implies that the PDE *velocity vector* $\frac{\partial \mathbf{I}(M)}{\partial t}$ must be orthogonal to the vector $\mathbf{I}(M)$, in order to preserve its norm. Note this is a pointwise constraint that doesn't depend on spatial relations between vector pixels. Consider then the following vector PDE of the general form :

$$\frac{\partial \mathbf{I}}{\partial t} = \boldsymbol{\beta} \qquad \text{where } \boldsymbol{\beta} \in \mathbb{R}^n$$

Adding the norm constraint can be naturally done by finding the component of the unconstrained velocity $\boldsymbol{\beta}$ that is orthogonal to the vector $\mathbf{I}$ (Figure 5) :

$$\boldsymbol{\beta}_c = \mathcal{P}_{\mathbf{I}}^{\perp}(\boldsymbol{\beta}) = \boldsymbol{\beta} - \left(\frac{\boldsymbol{\beta}\,.\,\mathbf{I}}{\|\mathbf{I}\|^2}\right)\,\mathbf{I}$$

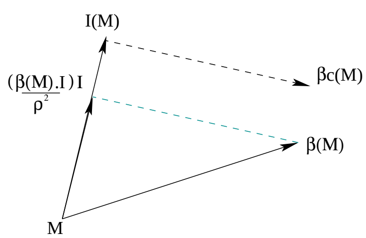


Figure 5: *Geometric view of the norm constraint*

Then, the following PDE ensures the preservation of the vector norm $\|\mathbf{I}\|$ at each point $M \in \Omega$:

$$\frac{\partial \mathbf{I}}{\partial t} = \boldsymbol{\beta} - \left(\frac{\boldsymbol{\beta}\,.\,\mathbf{I}}{\|\mathbf{I}\|^2}\right)\,\mathbf{I} \tag{14}$$

For the particular case of *vector field restorations under constrained norm*, we can choose $\boldsymbol{\beta}$ to be the expression of eq.(13). It allows the use of shock filters as well as accurate diffusion for norm constrained vector fields. Interesting applications of this equation are :

- Chromaticity restoration of noisy color image : It consists of splitting the color vectors $\mathbf{I} = (R, G, B)$ into the unit chromaticity vector $\mathbf{I}/\|\mathbf{I}\|$ and the brightness values $\|\mathbf{I}\|$, and apply a norm constrained PDE like eq.(14) on the obtained chromaticity vector image. If the noise is known to be chromatic, the obtained restoration will be more better since an adapted equation will be used.
- Direction regularization : This equation (14) is also able to restore 2D vector fields, coming for instance from optical flow calculation.

6 Applications

We present some results on vector image restoration (mainly *color images*, $n = 3$), using the proposed equations (7),(8),(9),(10), (11),(13),(14) in order to restore the following type of images :

- Color images with coupled noise (Figure 6a,b,c,d,e,f,g,h). The PDE's were applied on the (R, G, B) color basis. Note how the use of vector shock filters in our equation (13) (Figure 6f,h) allows to preserve the edges for a long time, durng the PDE flow.
- Color images with chromaticity noise : Figure 6i,j,k illustrates that norm preserving PDE's are better adapted to remove this kind of noise, and allows to preserve the little structures (look at the center of the flower for instance).
- Direction fields (Figure 6l,m) : The resulting directions are combed by the diffusion equation, while important discontinuities are preserved.
- Blurred color images (Figure 6n,o) : Our extension of the shock filter formulation can be used to enhance blurred images, without any knowledge of the cause of the degradation (instead of deconvolution methods).

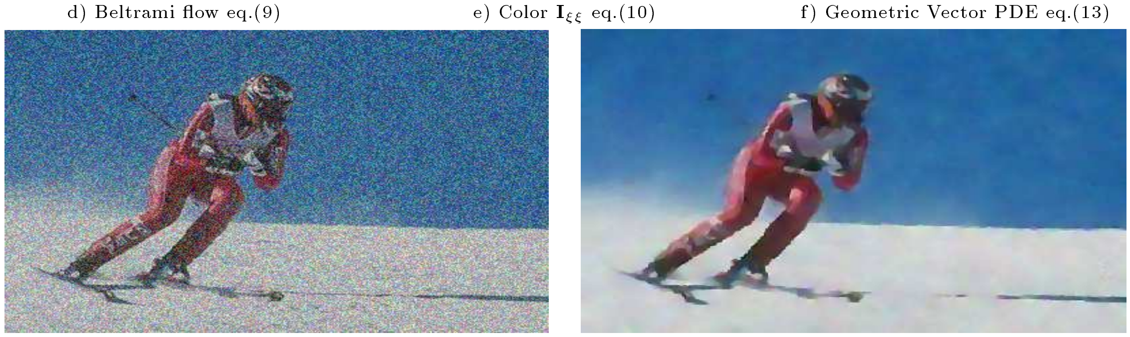
Note also that the authors web page *http://www-sop.inria.fr/robotvis/personnel* includes further results and PDE evolution movies.

Conclusion

In this paper, we proposed a local and geometric point of view of vector image filtering, using diffusion PDE's. It allowed us to analyze recent proposed methods of vector data regularization, as well as propose a new vector PDE, well adapted for image restoration. This equation, whose key feature is the use of a local vector geometry, combines the advantages of diffusion PDE's for noise removing, but also uses vector shock filters in order to enhance blurred edges. The extension to norm constrained vector fields can be the start for other well known constrained problems, as optical flow computation, orientation analysis, tensor image restoration. Promising results have been already obtained in recent papers [12, 55, 54, 56].

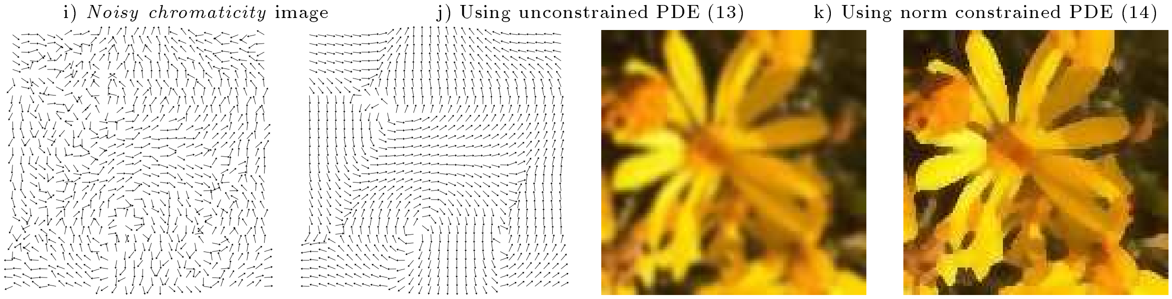
a) Noisy color image b) Color TV eq.(7) c) Coherence enhancing eq.(8)

d) Beltrami flow eq.(9) e) Color $\mathbf{I}_{\xi\xi}$ eq.(10) f) Geometric Vector PDE eq.(13)



g) Noisy color image h) Geometric Vector PDE eq.(13)

i) *Noisy chromaticity* image j) Using unconstrained PDE (13) k) Using norm constrained PDE (14)



l) Noisy direction field m) Restored field eq.(14) n) Blurred color image o) Vector shock filters eq.(12)

Figure 6: *Some results of vector-valued diffusion with PDE's* : comparative results (a,b,c,d,e,f,g,h), noisy chromaticity restoration (i,j,k), direction field regularization (l,m) and edge enhancing with vector shock filters (n,o).

References

[1] L. Alvarez, R. Deriche, J. Weickert, and J. Sànchez. Dense disparity map estimation respecting image discontinuities: A PDE and scale-space based approach. *International Journal of Visual Communication and Image Representation, Special Issue on Partial Differential Equations in Image Processing, Computer Vision and Computer Graphics, vol.13, p.3-21*, 2002.

[2] L. Alvarez, P.L. Lions, and J.M. Morel. Image selective smoothing and edge detection by nonlinear diffusion (II). *SIAM Journal of Numerical Analysis*, 29:845–866, 1992.

[3] M. Bertalmio, L.T. Cheng, S. Osher, and G. Sapiro. Variational problems and PDE's on implicit surfaces. *Proc. of VLSM'01, p.186-193, Vancouver/Canada*, July 2001.

[4] M. Bertalmio, G. Sapiro, V. Caselles, and C. Ballester. Image inpainting. *Proc. of SIGGRAPH 2000, p.417-424, New Orleans, USA*, July 2000.

[5] P. Blomgren and T.F. Chan. Color tv: Total variation methods for restoration of vector-valued images. *IEEE Trans. Imag. Proc.*, 7(3):304–309, 1998. Special Issue on Partial Differential Equations and Geometry-Driven Diffusion in Image Processing and Analysis.

[6] V. Caselles, J.M. Morel, G. Sapiro, and A. Tannenbaum. Introduction to the special issue on partial differential equations and geometry-driven diffusion in image processing and analysis. *IEEE Transactions on Image Processing*, 7(3):269–273, 1998.

[7] A. Chambolle and P.L. Lions. Image recovery via total variation minimization and related problems. *Nümerische Mathematik*, 76(2):167–188, 1997.

[8] T. Chan and J. Shen. Variational restoration of non-flat image features : Models and algorithms. *SIAM Journal in Applied Mathematics vol. 61(4), p.1338-1361*, 2000.

[9] T. Chan and J. Shen. Non-texture inpaintings by curvature-driven diffusions (cdd). Technical Report 00-35, Department of Mathematics, UCLA, Los Angeles, September 2000.

[10] P. Charbonnier, G. Aubert, M. Blanc-Féraud, and M. Barlaud. Two deterministic half-quadratic regularization algorithms for computed imaging. In *Proceedings of the International Conference on Image Processing*, volume II, pages 168–172, 1994.

[11] P. Charbonnier, L. Blanc-Féraud, G. Aubert, and M. Barlaud. Deterministic edge-preserving regularization in computed imaging. *IEEE Transactions on Image Processing*, 6(2):298–311, 1997.

[12] C. Chefd'hotel, D. Tschumperlé, R. Deriche, and O. Faugeras. Constrained flows on matrix-valued functions : application to diffusion tensor regularization. In *Proceedings of ECCV'02, p.251-265, Copenhagen/Denmark*, June 2002.

[13] L. Cohen. Auxiliary variables and two-step iterative algorithms in computer vision problems. *Journal of Mathematical Imaging and Vision, vol.6, p.59-83*, 1996.

[14] G.H. Cottet and L. Germain. Image processing through reaction combined with nonlinear diffusion. *Mathematics of Computation*, 61(204):659–673, October 1993.

[15] O. Coulon, D.C. Alexander, and S.R. Arridge. A regularization scheme for diffusion tensor magnetic resonance images. In *XVIth International Conference on Information Processing in Medical Imaging, vol 2082, p.92-105, David, California USA*, 2001.

[16] R. Deriche and O. Faugeras. Les EDP en Traitement des Images et Vision par Ordinateur. *Traitement du Signal*, 13(6), 1996.

[17] S. Geman and D.E. McClure. Bayesian image analysis: an application to single photon emission tomography. *Proc. of the Statistical Computing Section p.12-18, Washington D.C*, 1985

[18] A. Ben Hamza, G.R. Unal, and H. Krim. Towards a unified estimation theme: probabilistic vs. variational. *Signal Processing Magazine Special Issue*, 2002.
Mathematical methods for curves and surfaces II' Daehlen, Lyche, and Schumaker (eds.), Vanderbilt Univ. Press, ISBN 0-8265-1315-8, pages 271-278, 1998.

[19] R. Kimmel. A natural geometric norm for multi channel image processing. *Mathematical methods for curves and surfaces II' Daehlen, Lyche, and Schumaker (eds.), Vanderbilt Univ. Press, ISBN 0-8265-1315-8, p. 271-278,* 1998.

[20] R. Kimmel, R. Malladi, and N. Sochen. Images as embedded maps and minimal surfaces: movies, color, texture, and volumetric medical images. *International Journal of Computer Vision*, 39(2):111–129, September 2000.

[21] R. Kimmel and N. Sochen. Orientation diffusion or how to comb a porcupine. *Special issue on PDEs in Image Processing, Computer Vision, and Computer Graphics, Journal of Visual Communication and Image Representation, 13:238-248,* 2002.

[22] P. Kornprobst, R. Deriche, and G. Aubert. Image coupling, restoration and enhancement via PDE's. In *Proceedings of the International Conference on Image Processing*, volume 4, pages 458–461, Santa Barbara, California, October 1997.

[23] P. Kornprobst, R. Deriche, and G. Aubert. Nonlinear operators in image restoration. In *Proceedings of the International Conference on Computer Vision and Pattern Recognition*, pages 325–331, San-Juan, Puerto Rico, June 1997. IEEE Computer Society, IEEE.

[24] P. Kornprobst, R. Deriche, and G. Aubert. EDP, débruitage et réhaussement en traitement d'image: Analyse et contributions. In *11ème Congrés RFIA*, volume 1, pages 277–286. AFCET, January 1998.

[25] A. Koschan. A comparative study on color edge detection. In *Proceedings of the 2nd Asian Conference on Computer Vision*, volume 3, pages 574–578, Singapore, December 1995.

[26] Q.T. Luong. La couleur en vision par ordinateur: une revue. *Traitement du Signal*, 8(1):3–34, 1991.

[27] R. Malladi and J.A. Sethian. Image processing: Flows under min/max curvature and mean curvature. *Graphical Models and Image Processing*, 58(2):127–141, March 1996.

[28] J.M. Morel and S. Solimini. Segmentation of images by variational methods: A constructive approach. *Rev. Math. Univ. Complut. Madrid*, 1:169–182, 1988.

[29] D. Mumford and J. Shah. Optimal approximations by piecewise smooth functions and associated variational problems. *Communications on Pure and Applied Mathematics*, 42:577–684, 1989.

[30] N. Nordström. Biased anisotropic diffusion - a unified regularization and diffusion approach to edge detection. *Image and Vision Computing*, 8(11):318–327, 1990.

[31] S. Osher and L.I. Rudin. Feature-oriented image enhancement using shock filters. *SIAM Journal of Numerical Analysis*, 27(4):919–940, August 1990.

[32] A. Pardo and G. Sapiro. Vector probability diffusion. In *Proceedings of the International Conference on Image Processing, Vancouver/Canada*. IEEE Signal Processing Society, September 2000.

[33] P. Perona. Orientation diffusions. *IEEE Transactions on Image Processing*, 7(3):457–467, March 1998.

[34] P. Perona and J. Malik. Scale-space and edge detection using anisotropic diffusion. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 12(7):629–639, July 1990.

[35] I. Polyak. Segmentation and noise suppressionvia nonlinear multiscale filtering. *Signal Processing Magazine Special Issue*, 2002.

[36] C. Poupon, J.F Mangin, V. Frouin, J. Regis, F. Poupon, M. Pachot-Clouard, D. Le Bihan, and I. Bloch. Regularization of mr diffusion tensor maps for tracking brain white matter bundles. In W.M. Wells, A. Colchester, and S. Delp, editors, *Medical Image Computing and Computer-Assisted Intervention-MICCAI'98*, number 1496 in Lecture Notes in Computer Science, pages 489–498, Cambridge, MA, USA, October 1998. Springer.

[37] C.A. Poynton. Poynton's colour faq (www.inforamp.net/ poynton). Web page, 1995.

[38] M. Proesmans, E. Pauwels, and L. Van Gool. *Coupled Geometry-Driven Diffusion Equations for Low-Level Vision*, pages 191–228. Computational imaging and vision. Kluwer Academic Publishers, 1994.

[39] L. Rudin and S. Osher. Total variation based image restoration with free local constraints. In *Proceedings of the International Conference on Image Processing*, volume I, pages 31–35, 1994.

[40] L. Rudin, S. Osher, and E. Fatemi. Nonlinear total variation based noise removal algorithms. *Physica D*, 60:259–268, 1992.

[41] G. Sapiro. Vector-Valued Active Contours. In *Proceedings of the International Conference on Computer Vision and Pattern Recognition*, pages 680–685, San Francisco, CA, June 1996. IEEE.

[42] G. Sapiro. Color snakes. *Computer Vision and Image Understanding*, 68(2), p.247-253, 1997.

[43] G. Sapiro. *Geometric Partial Differential Equations and Image Analysis*. Cambridge University Press, 2001.

[44] G. Sapiro and V. Caselles. Contrast enhancement via image evolution flows. *Graphical Models and Image Processing*, 59(6):407–416, 1997.

[45] G. Sapiro and D.L. Ringach. Anisotropic diffusion of multivalued images with applications to color filtering. *IEEE Transactions on Image Processing*, 5(11):1582–1585, 1996.

[46] J. Shah. A common framework for curve evolution, segmentation and anisotropic diffusion. *International Conference on Computer Vision and Pattern Recognition, San Francisco, Ca, USA, p.136-142*, 1996.

[47] N. Sochen, R. Kimmel, and R. Malladi. From high energy physics to low level vision. Technical Report 39243, LBNL report, UC Berkeley, 1996.

[48] N. Sochen, R. Kimmel, and R. Malladi. A geometrical framework for low level vision. *IEEE Transaction on Image Processing, Special Issue on PDE based Image Processing*, 7(3):310–318, 1998.

[49] B. Tang, G. Sapiro, and V. Caselles. Direction diffusion. *International Conference on Computer Vision, p.1245-1252, Kerkya/Greece*, 1998.

[50] B. Tang, G. Sapiro, and V. Caselles. Diffusion of general data on non-flat manifolds via harmonic maps theory : The direction diffusion case. *The International Journal of Computer Vision*, 36(2):149–161, February 2000.

[51] A.N. Tikhonov. Regularization of incorrectly posed problems. *Soviet. Math. Dokl.*, 4:1624–1627, 1963.

[52] D. Tschumperlé and R. Deriche. Restauration d'images vectorielles par EDP. In *12ème Congrès RFIA'00*, volume II, pages 247–256, Paris, February 2000.

[53] D. Tschumperlé and R. Deriche. Constrained and unconstrained pde's for vector image restoration. In Ivar Austvoll, editor, *Proceedings of the 10th Scandinavian Conference on Image Analysis*, pages 153–160, Bergen, Norway, June 2001.

[54] D. Tschumperlé and R. Deriche. Diffusion tensor regularization with constraints preservation. In *IEEE Computer Society Conference on Computer Vision and Pattern Recognition*, Kauai Marriott, Hawaii, December 2001.

[55] D. Tschumperlé and R. Deriche. Regularization of orthonormal vector sets using coupled PDE's. In *IEEE Workshop on Variational and Level Set Methods*, pages 3–10, Vancouver, Canada, July 2001.

[56] D. Tschumperlé and R. Deriche. Régularisation par EDP de champs de vecteurs orthonormés et applications. In *13ème Congrès RFIA'02, p.869-878*, Angers/France, January 2002.

[57] B. Vemuri, Y. Chen, M. Rao, T. McGraw, T. Mareci, and Z. Wang. Fiber tract mapping from diffusion tensor mri. In *1st IEEE Workshop on Variational and Level Set Methods in Computer Vision, Vancouver/Canada, p.81-88*, July 2001.

[58] J. Weickert. *Anisotropic diffusion in image processing* ECMI Series, Teubner, Stuttgart, 1998.

[59] J. Weickert. Coherence-enhancing diffusion of colour images. *7th National Symposium on Pattern Recognition and Image AnalysisVII, NSPRIA, Barcelona, April 21-25, 1997), Vol. 1, 239-244*, April 1997.

[60] J. Weickert. A review of nonlinear diffusion filtering. *Scale-Space Theory in Computer Vision, Lecture Notes in Comp. Science (Springer, Berlin)*, 1252:3–28, 1997. Invited Paper.

[61] J. Weickert. *Anisotropic Diffusion in Image Processing*. Teubner-Verlag, Stuttgart, 1998.

[62] J. Weickert. Linear scale space has first been proposed in Japan. *Journal of Mathematical Imaging and Vision*, 10(3):237–252, May 1999.

[63] A. Yezzi. Modified curvature motion for image smoothing and enhancement. *IEEE Trans. on Image Processing, Special Issue on PDE based Image Processing*, 7(3):345–352, 1998.

[64] Y.L. You, M. Kaveh, W.Y. Xu, and A. Tannenbaum. Analysis and Design of Anisotropic Diffusion for Image Processing. In *Proceedings of the International Conference on Image Processing, Austin/Texas, USA*, volume II, pages 497–501, November 1994.

[65] S. Di Zenzo. A note on the gradient of a multi-image. *Computer Vision, Graphics, and Image Processing*, 33:116–125, 1986.

David Tschumperlé gradutated from *Ecole Supérieure en Sciences Informatiques*, Sophia Antipolis, in 1999. He is actually working towards a Computer Science Ph.D in the *ROBOTVIS/ ODYSSEE* lab at INRIA/Sophia-Antipolis. He is mainly interested in variational tools and PDE's to solve computer vision problems, particularly related to multi-valued images.

Web : *http://www.inria.fr/robotvis/personnel/David.Tschumperle/*
E-Mail : *David.Tschumperle@sophia.inria.fr*

Rachid Deriche graduated from *Ecole Nationale Supérieure des Télécommunications*, Paris, in 1979 and received the Ph.D degree in Mathematics from the University of Paris IX, Dauphine in 1982. He is currently a Research Director at INRIA Sophia-Antipolis in the Computer and Biological Vision Group. His research interests are in Image Processing, Computer and Biological Vision and include in particular the area related to variational methods and partial differential equations for vision. More generally, he is very interested by the applications of mathematics to Computer Vision and Image Processing. Hes has authored and co-authored more than 100 scientific papers. To find out more about his research and some selected publications, take a look at :

Web : *http:// www.inria.fr/robotvis/personnel/der/*
E-Mail : *der@sophia.inria.fr*